Fourier Series

Fourier Series corresponding to a function $f(x)$ in an interval $-\pi \leq x \leq \pi$ is the trigonometric Series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where,

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}\right\} \circledast$$

(To indicate that a Fourier series arises from a function $f(x)$, we use the symbol '$\sim$' which is read generates, thus

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, a_0, $\{a_n\}$, $\{b_n\}$ are constants determined from $\circledast$)

Piecewise monotone function-

A function f is called Piecewise monotone on $[a, b]$, if $\exists$ a partition P of $[a, b]$ s.t. f is monotone on each of the subintervals.

$\circledast$ Thus if 'f' be piecewise monotone & bounded on $[a, b]$, it can have discontinuities of the first kind only.

Dirichlet's Condition

A function $f(x)$ will be said to satisfy Dirichlet's condition on an interval $[-\pi, \pi]$ in which it is defined when it is subjected to one of the following two conditions

(i) $f(x)$ is bounded & periodic with periodicity 2π & integrable on $[-\pi, \pi]$ & piecewise monotonic on $[-\pi, \pi]$.

(ii) $f(x)$ has a finite no. of points of infinite discontinuity in the interval.

When arbitrary small nbds. of these points are excluded, $f(x)$ is bounded in the remainder of the interval & this can be broken up into a finite no. of open partial intervals, in each of which $f(x)$ is monotonic.

$x_1 \quad x_2 \quad x_3 \quad \dots \quad x_n$

Further $\int_{-\pi}^{\pi} f(x)\, dx$ is to be absolutely Conv.

Convergence.

* When $f(x)$ satisfies Dirichlet's Cond^n on $-\pi \leq x \leq \pi$, the Fourier Series corresponding to $f(x)$ converges to $f(x)$ at any point x on $-\pi < x < \pi$ when $f(x)$ is Cont. & converges to $\frac{1}{2}\{f(x+0) + f(x-0)\}$ when there is an ordinary discont. at the point.

In particular at $x = \pi$ & $x = -\pi$, it converges to $\frac{1}{2}\{f(-\pi+0) + f(\pi-0)\}$ when $f(-\pi+0)$ & $f(\pi-0)$ exist.

Odd & Even functions.

$f(-x) = f(x)$
$f(x) = x^2$

If $f(x)$ is odd function

$f(x)\cos nx$ is odd & $f(x)\sin nx$ is even function.

Then, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$

& $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx.$

In this case the series will be $\sum_{n=1}^{\infty} b_n \sin nx.$

If $f(x)$ is even function

Then, $f(x)\cos nx$ is even & $f(x)\sin nx$ odd fn.

Then, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$

In this case series will be

$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx.$

* Change of scale. (In $-l \le y \le l$)

Let $\phi(y)$ be integrable over $-l \le y \le l$.

Put, $y = \frac{lx}{\pi}$ & let, $f(x) = \phi\left(\frac{lx}{\pi}\right)$.

We know the fourier series corresponding to $f(x)$ is, $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$. Then the fourier series corresp. to $\phi(y)$ is,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi y}{l} + b_n \sin\frac{n\pi y}{l}\right)$$

where, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{l}\int_{-l}^{l}\phi(y)\cos\frac{n\pi y}{l}\,dy.$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{l}\int_{-l}^{l}\phi(y)\sin\frac{n\pi y}{l}\,dy.$

■ Parallel discussion for convergence. in $[-l, l]$

Change of scale (In $0 \leq x \leq 2\pi$)

If f be bounded periodic with period 2π & integrable on $[0, 2\pi]$ & piecewise monotone on $[0, 2\pi]$, then

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, $a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$$

■ ||le discussion for convergence in $[0, 2\pi]$.

Sine and Cosine series : The Interval 0 to π only.

$[-\pi, \pi]$

(A) Cosine series:

Defining $f(x)$ on $[-\pi, 0)$ by the equ^n $f(-x) = f(x)$, i.e. taking $f(x)$ as an even function, we can show that:

i.e. when $b_n = 0$

i.e. when $f(x)$ is even fn.

If $f(x)$ satisfies Dirichlet's cond^n on $[0, \pi]$, then the following series is the Fourier Cosine series corresponding to $f(x)$.

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

The series is equal to $\frac{1}{2} \{ f(x+0) + f(x-0) \}$ at every x on $0 < x < \pi$ where $f(x+0)$ & $f(x-0)$ both exist. The series is equal to $f(0+0)$ at $x=0$ and equal to $f(\pi - 0)$ at $x = \pi$, provided both $f(0+0)$ & $f(\pi - 0)$ exist.

If $f(x)$ be cont. in $[0, \pi]$, then the cosine series represents $f(x)$ in $[0, \pi]$

(B) Sine Series.

Defining $f(x)$ on $[-\pi, 0)$ by the equation $f(-x) = -f(x)$
i.e. taking $f(x)$ as an odd function, we can
show that: If $f(x)$ satisfies Dirichlet's condition
on $[0, \pi]$, then the following series is the
Fourier sine series corresponding to $f(x)$.

$$\sum_{n=1}^{\infty} b_n \sin nx \quad \text{where, } b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

The series is equal to . . . . . . both exist.

- - . . When $x = 0$ and $x = \pi$, the sum is zero.

Fourier series expansion in the interval $[-l, l]$.

We assume that the function $f(x)$ is piecewise cont in $[-l, l]$.
Using the substitution $x = \frac{ly}{\pi}$ $(-\pi \le x \le \pi)$, we can
transform it into the function
$F(y) = f\left(\frac{ly}{\pi}\right)$, which is defined & int on $[-\pi, \pi]$.
Fourier series expansion of $F(y)$ can be written as,

$$F(y) = f\left(\frac{ly}{\pi}\right) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos ny + b_n \sin ny)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\,dy, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y) \cos ny \, dy \quad \& \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y) \sin ny \, dy.$$

Now, putting $y = \frac{\pi x}{l}$, we obtain

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right), \text{ where,}$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\,dx$$

$$\& \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\,dx$$

Problems.

① Show that the Fourier series corresponding to x^2 on $[-\pi, \pi]$ is $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}$.

& hence deduce that,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Ans. Let, $f(x) = x^2$ in $[-\pi, \pi]$

Since, $f(x)$ is continuous in $[-\pi, \pi]$, it is bounded and integrable on $[-\pi, \pi]$.

Now, $f'(x) = 2x$ $\therefore$ $f'(x) > 0$ for $x > 0$ & $f'(x) < 0$ for $x < 0$

So, $f(x)$ is mon. increasing in $(0, \pi]$ & is mon. decreasing in $[-\pi, 0)$.

So, $f(x)$ is piecewise monotone in $[-\pi, \pi]$.

So, $f(x)$ satisfies Dirichlet's condition on $[-\pi, \pi]$.

So, the Fourier series corresponding to $f(x) = x^2$ is $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

where, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$ & $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$

$a_0 = \frac{1}{\pi}\int f(x)\, dx$.

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx$

$= \frac{1}{\pi}\cdot 2\int_0^{\pi} x^2\, dx$

$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$

$= \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2\pi^2}{3}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx$

$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$

$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi}$

$\int x^2 \cos nx\, dx$

$= x^2\int \cos nx\, dx - \int\left[2x\cdot\int \cos nx\, dx\right]dx$

$= \frac{x^2 \sin nx}{n} - \frac{2}{n}\int x \sin nx\, dx$

$= \frac{x^2 \sin nx}{n} - \frac{2}{n}\left[x\left(\frac{-\cos nx}{n}\right) - \int\left(-\frac{\cos nx}{n}\right)dx\right]$

$= \frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2}{n^2}\cdot\frac{\sin nx}{n}$

$= \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2} - 0\right]$

$= \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$

$= \begin{cases} \frac{4}{n^2}, & n = \text{even} \\ -\frac{4}{n^2}, & n = \text{odd} \end{cases}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx = 0$ $\quad$ ($\because$ $x^2 \sin nx$ is an odd fn.)

So, the series corresp. to $f(x)$ is

$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ $\quad$ $[\because b_n = 0]$

$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4\cos nx}{n^2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \cdot \frac{\cos nx}{n^2}$.

Since, $f(x) = x^2$ is continuous on $[-\pi, \pi]$

So, on $[-\pi, \pi]$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \cdot \frac{\cos nx}{n^2}$$

$$= \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} - \cdots\right]$$

At, $x = \pm\pi$, the sum of the series

$$= \frac{1}{2}\left\{f(-\pi+0) + f(\pi-0)\right\}$$

$$= \frac{1}{2}\left(\pi^2 + \pi^2\right) = \pi^2$$

$$\pi^2 = \frac{\pi^2}{3} + 4\left[-\frac{(-1)}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

or, $\pi^2 - \frac{\pi^2}{3} = 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$

or, $\frac{2\pi^2}{3} = 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$

or, $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \to (1)$

At, $x = 0$, the sum of the series,

$$= \frac{1}{2}\left\{f(0-0) + f(0+0)\right\} = 0.$$

$$0 = \frac{\pi^2}{3} + 4\left[-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

or, $-\frac{\pi^2}{3} = -4\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$

or, $\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \to (2)$

If we add the above two series given in ① & ② we get

$$\frac{\pi^2}{6}+\frac{\pi^2}{12}=2\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

or, $$\frac{2\pi^2+\pi^2}{12}=2\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

or, $$\frac{3\pi^2}{24}=1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

or, $$\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

② If $f(x)=-x$, for, $-\pi<x<0$
$\qquad\quad = 0$, for, $0<x<\pi$

then show that Fourier series corresponding to $f(x)$ on $-\pi<x<\pi$ is

$$\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{n}$$

Ans. We note that $f(x)$ is not defined at $x=0,\pi,-\pi$; Here we can define $f(x)$ in any manner.

For convenience, let us take $f(x)=0$, at $x=0,\pi$
$f(x)=-x$ at $x=-\pi$ and $[-\pi,\pi]$, is

Thus $f(x)$ being continuous on $[-\pi,\pi]$, is bounded and integrable there.
Further $f(x)$ is monotone on each of the open intervals $-\pi<x<0$ and $0<x<\pi$. Thus $f(x)$ satisfies Dirichlet's condition on $[-\pi,\pi]$.
So, the Fourier series corresponding to $f(x)$ is,

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -x\,dx + \int_{0}^{\pi} 0 \cdot dx\right\}$$

$$= \frac{1}{\pi}\left[-\frac{x^2}{2}\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left(+\frac{\pi^2}{2}\right) = +\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -x \cdot \cos nx\,dx + \int_{0}^{\pi} 0 \cdot \cos nx\,dx\right\}$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx$$

$$= -\frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0}$$

$$= -\frac{1}{\pi}\left[0 + \frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)$$

$$\int x\cos nx\,dx$$

$$= \frac{x \cdot \sin nx}{n} - \int \frac{\sin nx}{n}\,dx$$

$$= \frac{x\sin nx}{n} + \frac{\cos nx}{n^2}$$

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -x\sin nx\,dx + \int_{0}^{\pi} 0 \cdot \sin nx\,dx\right\}$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx$$

$$= -\frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{0}$$

$$= -\frac{1}{\pi}\left[0 - \left(+\frac{\pi\cos n\pi}{n} + 0\right)\right]$$

$$= +\frac{\cos n\pi}{n} = \frac{(-1)^n}{n}$$

$$\int x\sin nx\,dx$$

$$= \frac{x(-\cos nx)}{n} + \int \frac{\cos nx}{n}\,dx$$

$$= -\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}$$

So, the Fourier series corresponding to $f(x)$ is,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2}\right)\cos nx + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.$$

$$= \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-2).}{(2n-1)^2}.\cos(2n-1)x + \sum_{n=1}^{\infty}\frac{(-1)^n \sin nx}{n}$$

$n=1$
$(-1)^n - 1 = -2$
$n=2$
$(-1)^n - 1 = 0$
$n=3$
$(-1)^n - 1 = 2$

③ Obtain the Fourier Series in the interval $[-\frac{1}{2}, \frac{1}{2}]$ of the function f given by $f(x) = \begin{cases} x - [x] - \frac{1}{2}, & \text{when } x \text{ is not an integer} \\ 0, & \text{otherwise} \end{cases}$

$f(x+1) = (x+1) - [x+1] - \frac{1}{2}$
$= x + 1 - \{[x]+1\} - \frac{1}{2} = x - [x] - \frac{1}{2} = f(x)$

so, $f(x)$ is a periodic function with periodicity 1.

Now, $f(-x) = -x - [-x] - \frac{1}{2} = -\{x + [-x] + \frac{1}{2} + 1 - 1\}$
$= -\{x + [1-x] - \frac{1}{2}\} = -\{x - [x] - \frac{1}{2}\}$
$= -f(x)$

So, f is an odd function & so $a_n = 0 \ \forall n$.

$x = \frac{y}{2}$, $y = \frac{2\pi}{2}$ ... $\therefore x = 2\pi y$.

Now, $b_n = 2\int_{-1/2}^{1/2} f(x)\sin(2n\pi x)\,dx$

$= 4\int_0^{1/2}(x - [x] - \frac{1}{2})\sin(2n\pi x)\,dx$

$= 4\left[\int_0^{1/2} x\sin(2n\pi x)\,dx - \int_0^{1/2}[x]\sin(2n\pi x)\,dx - \frac{1}{2}\int_0^{1/2}\sin(2n\pi x)\,dx\right]$

$= 4\left[-\frac{x\cos 2n\pi x}{2n\pi}\Big|_0^{1/2} + \int_0^{1/2}\frac{\cos 2n\pi x}{2n\pi}dx + \frac{1}{2}\frac{\cos 2n\pi x}{2n\pi}\Big|_0^{1/2}\right]$

$= 4\left[-\frac{1}{2}\cdot\frac{\cancel{\cos n\pi}}{\cancel{2n\pi}} + \frac{\sin 2n\pi x}{(2n\pi)(2n\pi)}\Big|_0^{1/2} + \frac{1}{2}\left(\frac{\cancel{\cos n\pi}}{\cancel{2n\pi}} - \frac{1}{2n\pi}\right)\right]$

$= 4\left[0 - \frac{1}{4n\pi}\right] = -\frac{1}{n\pi}$

So, the F. series is $\sum_{n=1}^{\infty}\left(-\frac{1}{n\pi}\right)\sin(2n\pi x)$